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On the relations between statistical values along unstable periodic orbits in differential equation systems
微分方程式における不安定周期軌道上の統計量間の関係について

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1 Introduction

A chaotic dynamical system includes an infinite number of unstable periodic orbits in general. Therefore, in order to capture statistical values of chaos weighted average along a series of unstable periodic orbits (UPOs), which is sometimes called a cycle expansion, has been proposed and used to study low dimensional systems [1, 2, 5]. Recently, in some turbulence systems in fluid dynamics, it has been shown that even only a few UPOs with relatively low periods can capture mean properties of chaotic motions [6, 7]. For example, for the turbulent Couette flow of rather low Reynolds number of the full Navier-Stokes system, Kawahara and Kida [6] obtained a remarkable agreement of an averaged velocity profile along a single UPO with that along a chaotic orbit in phase space of turbulent Couette flows. In the above studies, it seems that only a few UPOs with relatively low periods are enough to capture some mean properties of a chaotic solution. However, on the other hand, it appears that an UPO with longer period gives a better approximation to the statistical properties of chaotic solutions. So we may have a question why in the above systems even a small number of UPOs with rather low periods can give a remarkably good approximation to the various chaotic mean values. Some studies have been concerned with this problem. For example, Saiki and Yamada [10] employed three chaotic systems described by low dimensional ODEs and investigated the difference between the average of a dynamical quantity along an UPO and that along a chaotic orbit, especially with an attention focused on the dependence of the variance of the averaged value on the period of the UPOs. It was found that for each chaotic system the distributions of a time average of a dynamical variable along UPOs with lower and higher periods are similar to each other and the variance of the distribution is small, in contrast with that along chaotic segments. In this paper, we employ two chaotic systems described by differential equation systems and investigate the relation between the averages of dynamical quantities along UPOs detected numerically.

2 Relation between statistical values along UPOs

2.1 Lorenz system

UPOs in the Lorenz system ($dx/dt = \sigma(y-x), dy/dt = rx - y - xz, dz/dt = xy - bz$) with the classical parameter values ($\sigma = 10$, $b = 8/3$, $r = 28$) have been extensively studied [11, 4, 3, 14]. Although the Lorenz system is not uniformly hyperbolic, it is recently proved by the aid of numerical calculation with guaranteed accuracy that the Lorenz attractor is chaotic and includes an infinite number of UPOs densely [12, 13]. Here we focus our attention to the relations between time averaged values of dynamical variables along UPOs of the Lorenz system. In order to detect UPOs we employ in this paper the Newton-Raphson-Mees method in which the period of the UPO is regarded as a variable to be found in the numerical calculation. We found more than 1000 UPOs of the periods from 1.558 through 16.445, corresponding respectively from 2 through 23 rotations around a wing of the Lorenz attractor. The number of rotations corresponds to the period (PERIOD N) of the Poincaré map defined by the Poincaré section $z = r - 1, dz/dt > 0$. Figure 1 shows the relation of time average of z ($\langle z \rangle$) and the variance of z (Var of z) along UPOs ($N = 11$), along segments of chaotic orbits (chaotic segments) ($N = 1, \cdots, 11$) and along a long chaotic orbit. It shows that whereas the chaotic segments give scattering points in ($\langle z \rangle$, Var of z)-plane, the UPOs give an almost straight line. In addition, a chaotic orbit gives a point on the line. From Figure 2 we can see that also in (λ,$\langle z \rangle$)-plane the UPOs give points on a straight line, where λ represents the Lyapunov exponent. Figures 1 and 2 show that if we choose a special UPO whose Lyapunov exponent approximates that of a long chaotic orbit, the UPO also gives various macroscopic statistical quantities of chaos, even if the period is not large enough. This result implies that there are UPOs which can capture various types of macroscopic statistical quantities of chaos.

2.2 Kuramoto-Sivashinsky system

The Kuramoto-Sivashinsky (KS) system can be written as $u_t = -u_{xx} - \nu u_{xxxx} - (u^2)_x$, where the "viscosity" control parameter ν is fixed as 0.02991. We assume periodic boundary conditions $u(x,t) = u(x+2\pi, t)$ and through a symmetry reduction $u(x,t) = -(i/2)\sum a_k(t)e^{ikx}$ we obtain the equations in Fourier space [9]:

$$\dot{a}_k = (k^2 - \nu k^4)a_k + \frac{k}{2}\left(\sum_{m=k-N}^{-1} a_{-m}a_{k-m} - \sum_{m=1}^{k-1} a_m a_{k-m} + \sum_{m=k+1}^{N} a_m a_{m-k}\right),$$

$$k = 1, \cdots, N.$$

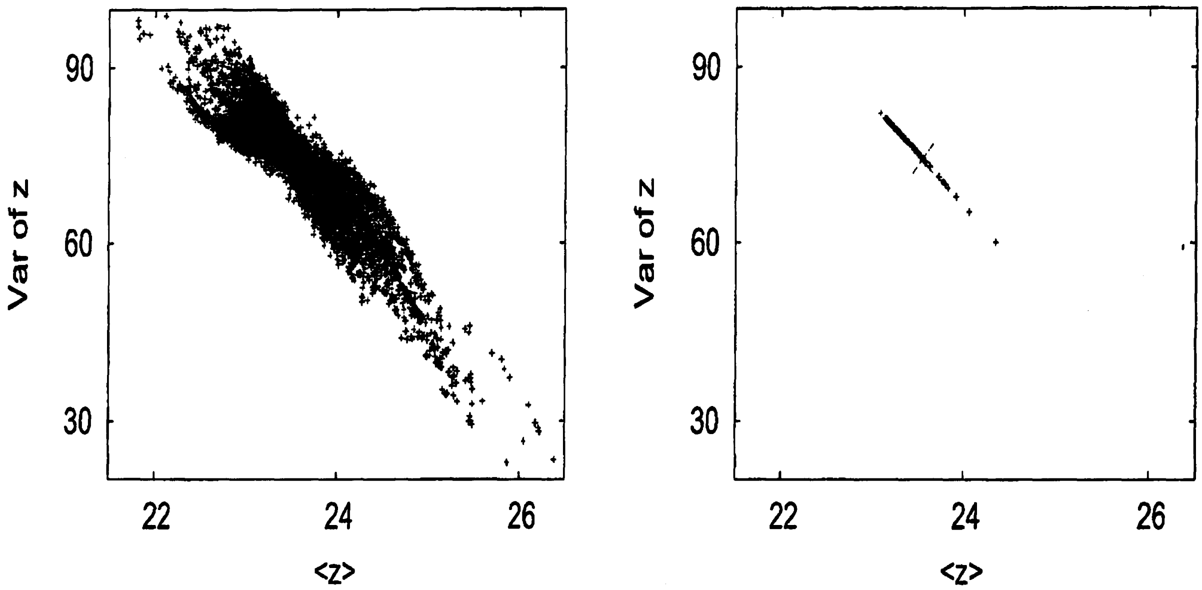


Fig. 1: ($\langle z \rangle$,Var of z) for each chaotic segment ($N = 11$) (small plus)(left) and each UPO ($N = 1, \cdots, 11$) (small plus) with chaos averages (big cross)(right) for the Lorenz system.

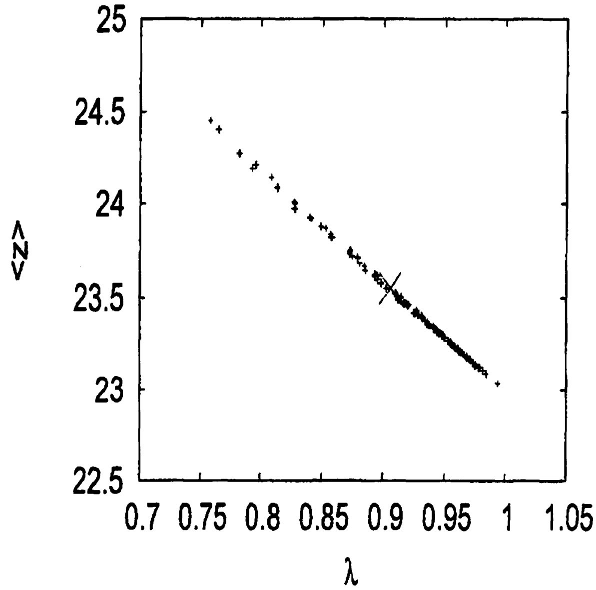
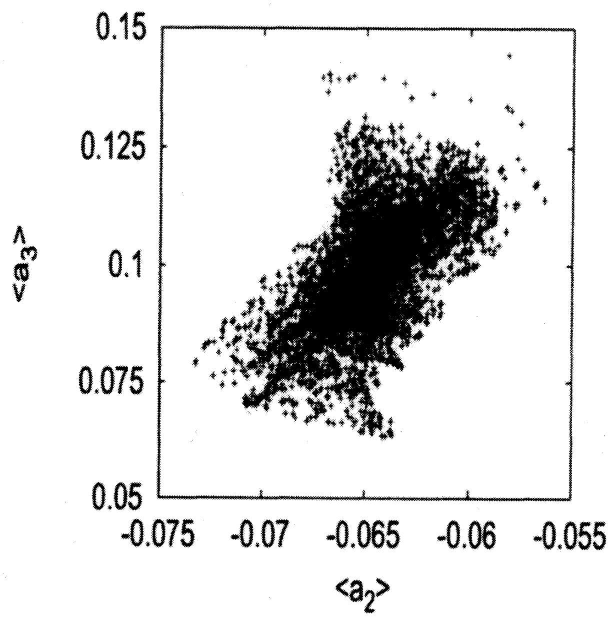


Fig. 2: ($\lambda, \langle z \rangle$) for each UPO ($N = 1, \cdots, 11$) (small plus) with chaos averages (big cross) for the Lorenz system.



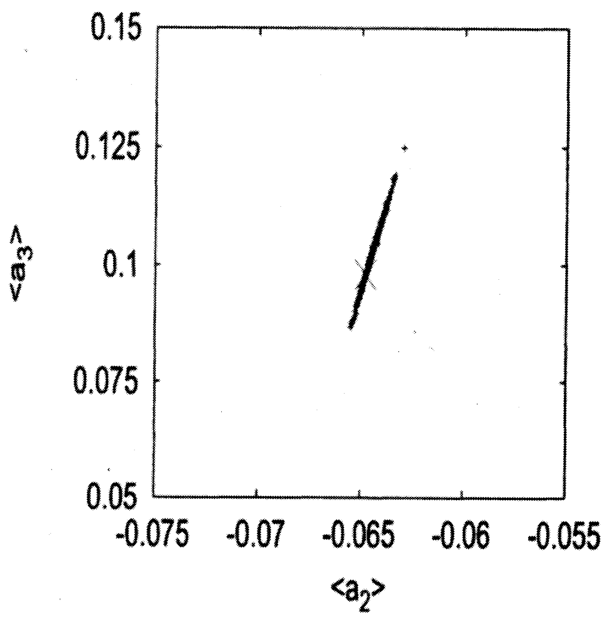


Fig. 3: $(\langle a_2\rangle, \langle a_3\rangle)$ for each chaotic segment ($N = 8$) (small plus)(left) and each UPO ($N = 1, \cdots, 12$) (small plus) and with chaos averages (big cross)(right) for the KS system.

N is fixed as 16 in this study. For the case of the KS system we obtain a similar result on the relation between time averaged values along UPOs. We see in Figure 3 that $(\langle a_2\rangle, \langle a_3\rangle)$ for each UPO with PERIOD 8 (Poincaré map is defined by the section $a_1 = 0, da_1/dt > 0$) is on a single almost straight line, whereas those along chaotic segments scatter in the plane. We should also remark that averages along a chaotic orbit are on the line.

3 Concluding Remarks

By employing two chaotic systems described by differential equation systems, we study relations between time averaged values of some variables along UPOs, and find that there is almost a one to one and linear correspondence. In addition, statistical values along a long chaotic orbit are on the same constraint as those along UPOs. This implies that UPO which approximates one of the statistical quantities along a long chaotic orbit approximates another statistical quantity, and suggests that longer UPOs of two systems can be approximable by some UPOs with low periods at least in the statistical sense. Moreover, we found in the Lorenz system that when the system possesses a tangency structure as a parameter changes, another class of UPOs appear outside the linear correspondence discussed above.

References

[1] T. Kai and K. Tomita, Statistical Mechanics of Deterministic Chaos, *Prog. Theor. Phys.* **64** (1980) 1532.

[2] P. Cvitanović, Invariant measurement of strange sets in terms of cycles, *Phys. Rev. Lett.* **61** (1988) 2729.

[3] B. Eckhardt and E. Ott, Periodic orbit analysis of the Lorenz attractor, *Zeitschrift fur Physik B Condensed Matter*, **93** (1994), 258.

[4] V. Franceschini, C. Giberti and Z. Zheng, Characterization of the Lorenz attractor by unstable periodic orbits, *Nonlinearity*, **6** (1993) 251–258.

[5] C. Grebogi, E. Ott and J. A. Yorke, Unstable periodic orbits and the dimensions of multifractal chaotic attractors, *Phys. Rev. A* **37** (1988) 1711.

[6] G. Kawahara and S. Kida, Periodic motion embedded in plane Couette turbulence, *J. Fluid Mech.*, **449** (2001) 291–300.

[7] S. Kato and M. Yamada, Unstable periodic solutions embedded in a shell model turbulence, *Phys. Rev. E* **68** (2003) 025302(R).

[8] L. van Veen, S. Kida and G. Kawahara, Periodic motion representing isotropic turbulence, *Fluid Dyn. Res.* **38** (2006) 19–46.

[9] E. L. Rempel and A. C.-L. Chian, Intermittency induced by attractor-merging crisis in the Kuramoto-Shivashinsky equation, *Phys. Rev. E* **71** (2005) 016203.

[10] Y. Saiki and M. Yamada, Time averaged properties of unstable periodic orbits and chaotic orbits in ordinary differential equation systems, *Phys. Rev. E* **38** (2009) 015201(R).

[11] C. Sparrow, The Lorenz equations: Bifurcations, chaos, and strange attractors. (Applied Mathematical Sciences, Vol.41), Springer-Verlag (1982).

[12] W. Tucker, The Lorenz attractor exists, *Comptes Rendus de l'Académie des Sciences. Série I. Mathmatique*, **328**(12) (1999), 1197–1202.

[13] W. Tucker, A Rigorous ODE Solver and Smale's 14th Problem, *Found. Comput. Math.*, **2**, (2002), 53–117.

[14] S. M. Zoldi, Unstable periodic orbit analysis of histograms of chaotic time series, *Physical Review Letters*, **81** (1998) (16) 3375.